

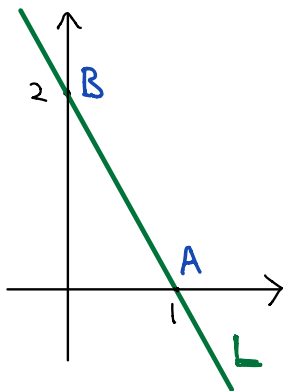
Math 2010 Week 2 (Au 1.3)

Linear Objects in \mathbb{R}^n

eg lines, plane, k-plane, hyperplane

Line

eg in \mathbb{R}^2



Equation form

$$2x + y = 2$$

Parametric form

$$\begin{aligned}(x, y) &= \vec{OA} + t \vec{AB} \\ &= (1, 0) + t(-1, 2) \\ &= (1-t, 2t)\end{aligned}$$

Varying $t \in \mathbb{R}$ gives
all the points on L

Symmetric / Slope form

Note that

$$\begin{cases} x = 1-t \\ y = 2t \end{cases} \Rightarrow \begin{cases} t = \frac{x-1}{-1} \\ t = \frac{y}{2} \end{cases} \Rightarrow \frac{x-1}{-1} = \frac{y}{2}$$

Parametric form of a line in \mathbb{R}^n

let L be a line in \mathbb{R}^n

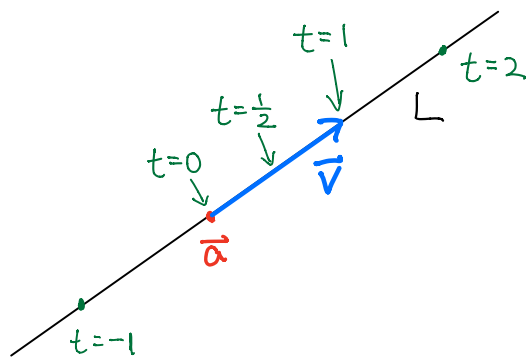
\vec{a} is a point on L

\vec{v} is a direction vector of L

Parametric form of L

$$\vec{x} = \vec{a} + t \vec{v}$$

$t \in \mathbb{R}$ is called a parameter



L is parametrized by $t \in \mathbb{R}$

eg A line L in \mathbb{R}^3 passes through

$$A=(1,2,3), B=(-1,3,5).$$

Then we can take $\vec{a}=(1,2,3)$,

$$\vec{v}=\overrightarrow{AB}=(-1-1, 3-2, 5-3)=(-2,1,2)$$

\therefore Parametric form:

$$(x,y,z)=(1,2,3)+t(-2,1,2) \quad (*)$$

Rmk ① Parametric form is not unique.

Other parametrizations of L

$$(x,y,z)=(-1,3,5)+t(2,-1,-2)$$

$$\text{or } (-1,3,5)+t(-4,2,4)$$

② From $(*)$, we get symmetric form

$$\frac{x-1}{-2} = \frac{y-2}{1} = \frac{z-3}{2} \quad (=t)$$

$$\Leftrightarrow \begin{cases} x-1 = -2(y-2) \\ x-1 = -(z-3) \end{cases}$$

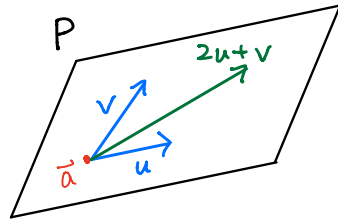
Planes in \mathbb{R}^3

A plane P in \mathbb{R}^3 can be determined by

- ① 3 non-colinear points on P ; or
- ② A point on P and 2 linearly independent directions (not same or opposite); or
- ③ A point on P and a normal vector

For ②

Let $\vec{a} \in P$, \vec{u}, \vec{v} are lin. indept directions of P



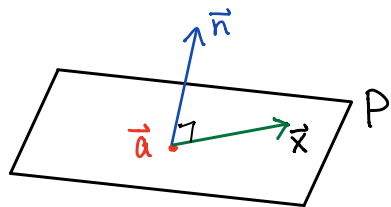
Parametric form of P

$$\vec{x} = \vec{a} + s\vec{u} + t\vec{v}$$

↑ a point on P ↑ parameters ↑ directions

For ③, suppose

- $\vec{a} = (a_1, a_2, a_3)$ is a point on P
- $\vec{n} = (n_1, n_2, n_3)$ is a normal vector of P



Let $\vec{x} = (x, y, z) \in \mathbb{R}^3$. Then

$$\begin{aligned}\vec{x} \text{ is on } P &\Leftrightarrow \vec{x} - \vec{a} \perp \vec{n} \\ &\Leftrightarrow (\vec{x} - \vec{a}) \cdot \vec{n} = 0 \\ &\Leftrightarrow \vec{x} \cdot \vec{n} = \vec{a} \cdot \vec{n}\end{aligned}$$

Equation of P

$$n_1x + n_2y + n_3z = \underbrace{a_1n_1 + a_2n_2 + a_3n_3}_{\text{constant}}$$

Rmk If $(a, b, c) \neq \vec{0}$, the equation

$$ax + by + cz = d$$

describes a plane in \mathbb{R}^3 with normal vector (a, b, c) .

eg Suppose P is a plane passing through

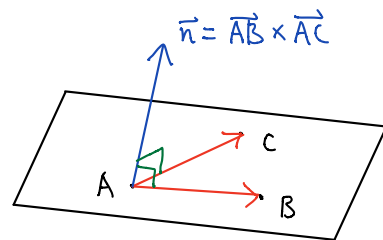
$$A = (0, 0, 1), B = (0, 2, 0) \quad C = (-1, 1, 0)$$

Represent P using (i) parametric form; (ii) equation

Sol (i)

$$\begin{aligned}\vec{AB} &= (0, 2, 0) - (0, 0, 1) \\ &= (0, 2, -1)\end{aligned}$$

$$\begin{aligned}\vec{AC} &= (-1, 1, 0) - (0, 0, 1) \\ &= (-1, 1, -1)\end{aligned}$$



\therefore Parametric form:

$$(x, y, z) = (0, 0, 1) + s(0, 2, -1) + t(-1, 1, -1)$$

$$(ii) \text{ Take } \vec{n} = \vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 2 & -1 \\ -1 & 1 & -1 \end{vmatrix} = (-1, 1, 2) \perp P$$

\therefore Equation for P :

$$[(x, y, z) - (0, 0, 1)] \cdot (-1, 1, 2) = 0$$

$$(-1)x + (1)y + 2(z - 1) = 0$$

$$-x + y + 2z = 2$$

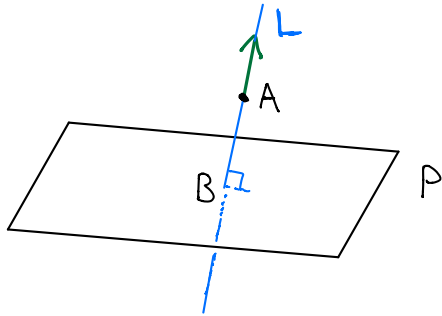
eg Find the distance between $A=(2,1,1)$
and the plane $P: -x+2y-z=-4$ *

Sol From *, $\vec{n} = (-1, 2, -1) \perp P$

Consider the line L

$$\vec{X}(t) = \vec{A} + t\vec{n} = (2, 1, 1) + t(-1, 2, -1)$$

Let $B = L \cap P$ be the intersection of L and P



Then B is the point on P closest to A

To find B , put

$$\vec{X}(t) = (2-t, 1+2t, 1-t) \text{ into } *$$

$$\begin{aligned} \text{Then } -(2-t) + 2(1+2t) - (1-t) &= -4 \\ \Rightarrow 6t - 1 &= -4 \Rightarrow 6t = -3 \Rightarrow t = -\frac{1}{2} \end{aligned}$$

$$\therefore B = \vec{X}\left(-\frac{1}{2}\right) = \left(\frac{5}{2}, 0, \frac{3}{2}\right)$$

Distance between A and P

$$\begin{aligned} = \|\vec{AB}\| &= \sqrt{\left(\frac{5}{2} - 2\right)^2 + (0 - 1)^2 + \left(\frac{3}{2} - 1\right)^2} \\ &= \frac{\sqrt{6}}{2} \end{aligned}$$

Ex Find the distance between the lines

$$L_1(s) = (-4, 9, -4) + s(4, -3, 0)$$

$$L_2(t) = (5, 2, 10) + t(4, 3, 2)$$

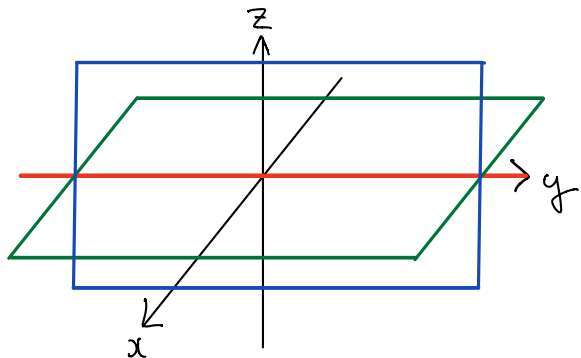
Hint: Find A on L_1 , B on L_2 such that

$$\vec{AB} \perp L_1, L_2$$

(Ans: 13)

Line in \mathbb{R}^3 by equations

eg1 y -axis in \mathbb{R}^3



One equation can only represent a plane.
We need at least two to represent a line.

$$\text{eg. } \begin{cases} x = 0 \\ z = 0 \end{cases} \leftrightarrow y\text{-axis}$$

System of equations \leftrightarrow Intersection of planes

Two ways to represent linear objects

Equation(s) $\xrightarrow{\text{Gaussian Elimination}}$ Parametric form
 $\xleftarrow{\text{Eliminate parameters}}$

eg2

Given equations of a line L

$$\begin{cases} x + y + 6z = 6 \\ x - y - 2z = -2 \end{cases} \longrightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ -4 \\ 1 \end{bmatrix}$$

(*)

Conversely, from (*)

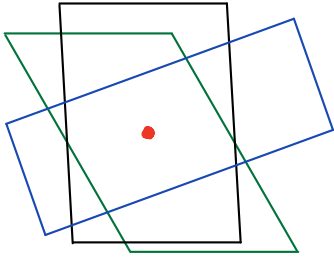
$$\begin{cases} x = 2 - 2t \\ y = 4 - 4t \\ z = t \end{cases} \Rightarrow \begin{cases} 2x - y = 0 \\ y + 4z = 4 \end{cases}$$

\therefore Another set of equations for L : $\begin{cases} 2x - y = 0 \\ y + 4z = 4 \end{cases}$

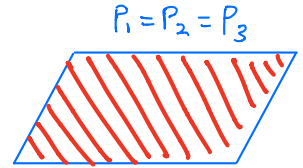
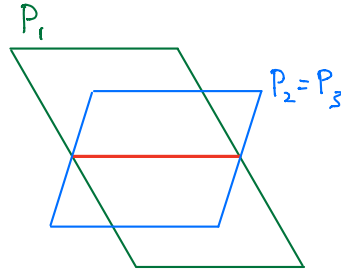
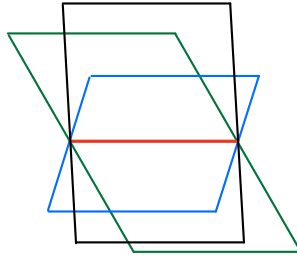
eg 3 Three non-trivial equations of the form $ax+by+cz=d$ (in \mathbb{R}^3)
means $(a,b,c) \neq (0,0,0)$

Pictures

Case 1: Unique solution

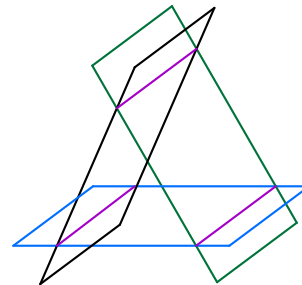
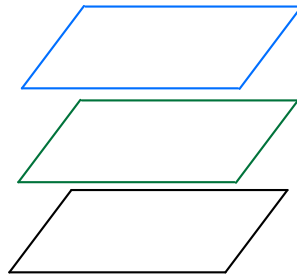
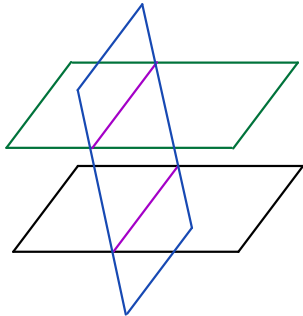


Case 2: Infinitely many solutions



Case 3: No solution \leftrightarrow No common intersection

eg



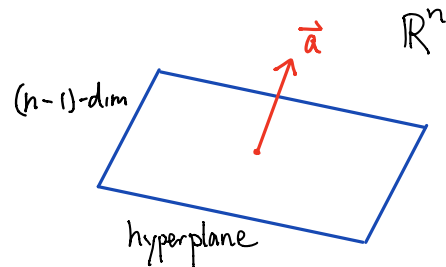
General linear objects in \mathbb{R}^n

- Similar to lines in \mathbb{R}^3 , we need a system of equations to describe a plane in \mathbb{R}^n
- In \mathbb{R}^n , an equation of the form

$$\vec{a} \cdot \vec{x} = a_1 x_1 + a_2 x_2 + \dots + a_n x_n = c, \quad \vec{a} \neq \vec{0}$$

describes a hyperplane (dimension = $n-1$) with normal vector \vec{a}

- To describe a k -dimensional plane P (called k -plane) in \mathbb{R}^n :



Parametric form

$$\vec{x} = \vec{q} + \sum_{i=1}^k t_i \vec{v}_i, \text{ where}$$

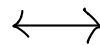
- $\vec{q} \in P$
- $\vec{v}_1, \dots, \vec{v}_k$ are k lin. indept vectors $\parallel P$
- t_1, \dots, t_k are parameters

Equations

\iff rank $A = n-k$
 $n-k$ non-redundant equations ($i=1, 2, \dots, n-k$)

$$\sum_{j=1}^n a_{ij} x_j = c_i$$

System of
 $n-k$ equations



Intersection of
 $n-k$ hyperplanes

Thm Let $\vec{x}(t) = (x_1(t), x_2(t), \dots, x_n(t))$. Then

$$\textcircled{1} \lim_{t \rightarrow a} \vec{x}(t) = \left(\lim_{t \rightarrow a} x_1(t), \dots, \lim_{t \rightarrow a} x_n(t) \right)$$

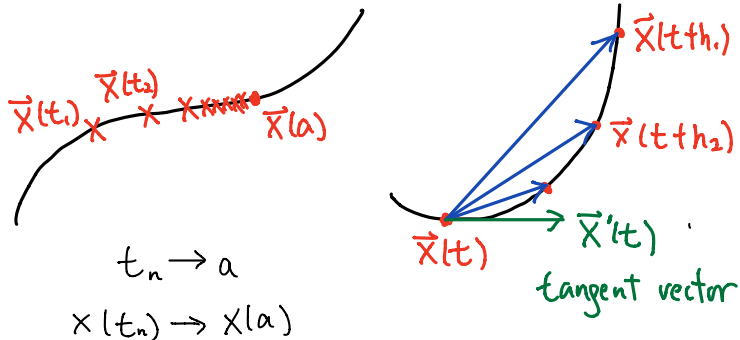
$$\textcircled{2} \vec{x}'(t) \stackrel{\text{Defn}}{=} \lim_{h \rightarrow 0} \frac{\vec{x}(t+h) - \vec{x}(t)}{h}$$

$$= (x_1'(t), x_2'(t), \dots, x_n'(t))$$

provided the limits exist.

Rmk $\vec{x}'(a) =$ tangent vector of $\vec{x}(t)$ at $t=a$.

Picture



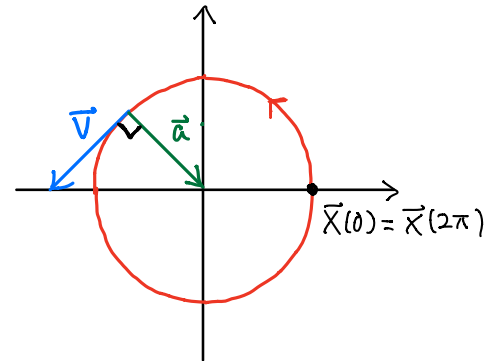
Physics

If $\vec{x}(t) =$ displacement at time t

Then $\vec{x}'(t) =$ velocity $= \vec{v}(t)$

$\vec{x}''(t) =$ acceleration $= \vec{a}(t)$

eg $\vec{x}(t) = (\cos t, \sin t)$ $0 \leq t \leq 2\pi$



$$\vec{v}(t) = \vec{x}'(t) = (-\sin t, \cos t) \perp \vec{x}(t)$$

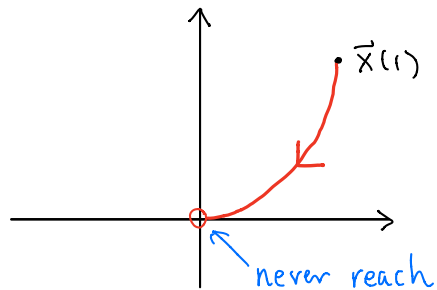
$$\vec{a}(t) = \vec{x}''(t) = (-\cos t, -\sin t) = -\vec{x}(t)$$

Also speed $= \|\vec{v}(t)\| = 1$

eg $\vec{x}: [1, \infty) \rightarrow \mathbb{R}^2$

$$\vec{x}(t) = \left(\frac{1}{t}, \frac{1}{t^2} \right)$$

$\underset{x}{\parallel} \quad \underset{y}{\parallel} \Rightarrow y = x^2$



$$\lim_{t \rightarrow \infty} \vec{x}(t) = \left(\lim_{t \rightarrow \infty} \frac{1}{t}, \lim_{t \rightarrow \infty} \frac{1}{t^2} \right)$$
$$= (0, 0)$$

Rules

Let $\vec{x}(t), \vec{y}(t)$ be curves in \mathbb{R}^n , $c \in \mathbb{R}$ be a constant
 $f(t)$ be a real-valued function.

① $(\vec{x}(t) \pm \vec{y}(t))' = \vec{x}'(t) \pm \vec{y}'(t)$

② $(c \vec{x}(t))' = c \vec{x}'(t)$

③ $(f(t) \vec{x}(t))' = f'(t) \vec{x}(t) + f(t) \vec{x}'(t)$

④ $(\vec{x}(t) \cdot \vec{y}(t))' = \vec{x}'(t) \cdot \vec{y}(t) + \vec{x}(t) \cdot \vec{y}'(t)$

⑤ If $n=3$,

$$(\vec{x}(t) \times \vec{y}(t))' = \vec{x}'(t) \times \vec{y}(t) + \vec{x}(t) \times \vec{y}'(t)$$

Rmk ③ - ⑤ are product rules

Arc length

Let $\vec{x}(t)$ be a curve with
 $\vec{x}'(t)$ exists and continuous

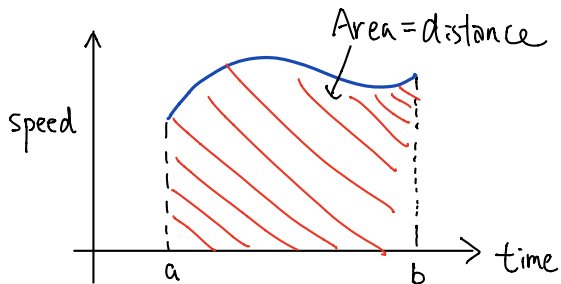
Arc length of $\vec{x}(t)$ for $a \leq t \leq b$ is

$$s = \int_a^b \|\vec{x}'(t)\| dt$$

If $\vec{x}(t)$ = displacement at time t
then $\vec{x}'(t)$ = velocity

$\|\vec{x}'(t)\|$ = speed

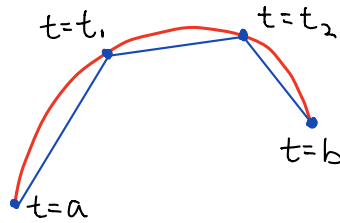
$\int_a^b \|\vec{x}'(t)\| dt$ = distance travelled



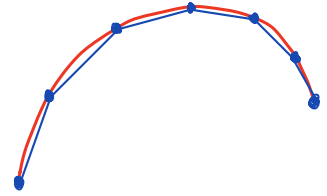
From Point of view of Mathematics

Approximate a curve by line segments:

Take $a = t_0 < t_1 < t_2 < \dots < t_n = b$



Approximation



Better approximation

$$s \approx \sum_{i=1}^n \|\vec{x}(t_i) - \vec{x}(t_{i-1})\|$$

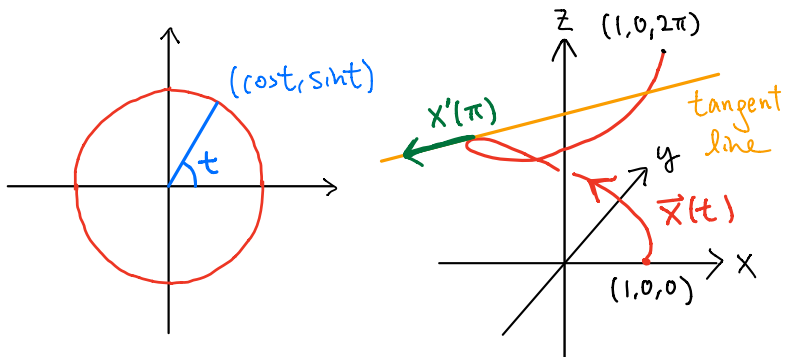
$$\approx \sum_{i=1}^n \|\vec{x}'(t_i)\| (t_i - t_{i-1})$$

(Recall: $x'(t_i) = \lim_{t \rightarrow t_i} \frac{x(t) - x(t_i)}{t - t_i}$)

Take Limit $\Rightarrow s = \int_a^b \|\vec{x}'(t)\| dt$

eg (Helix)

$$\vec{x}(t) = (\cos t, \sin t, t), t \in [0, 2\pi]$$



a. Find the tangent line of \vec{x} at $t = \pi$.

b. Find arclength of the helix.

Sol

a. $\vec{x}(t) = (\cos t, \sin t, t)$

$$\vec{x}'(t) = (-\sin t, \cos t, 1)$$

$$\vec{x}'(\pi) = (0, -1, 1) \leftarrow \text{direction of tangent}$$

Also, $\vec{x}(\pi) = (-1, 0, \pi) \leftarrow$ a point on tangent line

\therefore Parametric form of tangent line

$$\vec{x} = (-1, 0, \pi) + t(0, -1, 1)$$

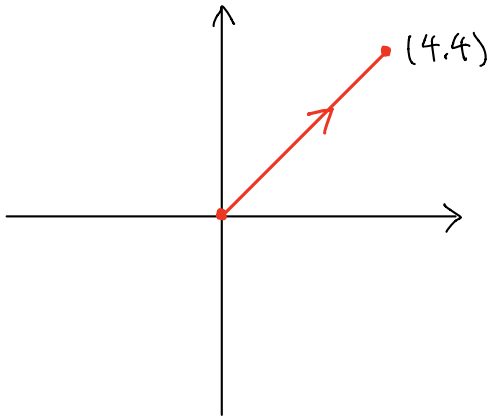
b. $\|\vec{x}'(t)\| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2}$
 $= \sqrt{2}$

$$\begin{aligned} \Rightarrow S &= \int_0^{2\pi} \|\vec{x}'(t)\| dt \\ &= \int_0^{2\pi} \sqrt{2} dt \\ &= [\sqrt{2}t]_0^{2\pi} \\ &= 2\sqrt{2}\pi \end{aligned}$$

Arc length is independent of parametrization

eg $\vec{x}(t) = (t, t) \quad 0 \leq t \leq 4$

$\vec{y}(t) = (t^2, t^2) \quad 0 \leq t \leq 2$



\vec{x}, \vec{y} are two parametrizations
of the same line segment

$$\vec{x}'(t) = (1, 1)$$

$$\vec{y}'(t) = (2t, 2t)$$

arclength of $\vec{x}(t)$

arclength of $\vec{y}(t)$

$$= \int_0^4 \|\vec{x}'(t)\| dt$$

$$= \int_0^2 \|\vec{y}'(t)\| dt$$

$$= \int_0^4 \sqrt{2} dt$$

$$= \int_0^2 \sqrt{(2t)^2 + (2t)^2} dt$$

$$= 4\sqrt{2}$$

$$= \int_0^2 2\sqrt{2} t dt$$

same
answer

$$= \left[\sqrt{2} t^2 \right]_0^2$$

$$= 4\sqrt{2}$$

Rmk

To prove arclength is independent of parametrization
use change of variable for integration (Ex)